

Axiomatization of Boolean algebras via weak dicomplementations

Léonard Kwuida

Université du Québec en Outaouais
Gatineau, Canada
leonard.kwuida@uqo.ca

Abstract. In this note we give an axiomatization of Boolean algebras based on weakly dicomplemented lattices: an algebra $(L, \wedge, \vee, \triangleleft)$ of type $(2, 2, 1)$ is a Boolean algebra iff (L, \wedge, \vee) is a non empty lattice and $(x \wedge y) \vee (x \wedge y^\triangleleft) = (x \vee y) \wedge (x \vee y^\triangleleft)$ for all $x, y \in L$. This provides a unique equation to encode distributivity and complementation on lattices.

1 Weakly dicomplemented lattices

Definition 1 ([Kw04]). *A weakly dicomplemented lattice is a bounded lattice L equipped with two unary operations \triangleleft and \triangleright called **weak complementation** and **dual weak complementation**, and satisfying for all $x, y \in L$ the conditions:*

- | | |
|--|---|
| (1) $x^{\triangleleft\triangleleft} \leq x$, | (1') $x^{\triangleright\triangleright} \geq x$, |
| (2) $x \leq y \implies x^\triangleleft \geq y^\triangleleft$, | (2') $x \leq y \implies x^\triangleright \geq y^\triangleright$, |
| (3) $(x \wedge y) \vee (x \wedge y^\triangleleft) = x$, | (3') $(x \vee y) \wedge (x \vee y^\triangleright) = x$. |

We call x^\triangleleft the **weak complement** of x and x^\triangleright the **dual weak complement** of x . The pair $(x^\triangleleft, x^\triangleright)$ is called the **weak dicomplement** of x and the pair $(\triangleleft, \triangleright)$ a **weak dicomplementation** on L . The structure $(L, \wedge, \vee, \triangleleft, 0, 1)$ is called a **weakly complemented lattice** and $(L, \wedge, \vee, \triangleright, 0, 1)$ a **dual weakly complemented lattice**.

Note that $x^{\triangleleft\triangleleft} \leq x \iff x^{\triangleleft\triangleleft} \vee x = x$ and $x^{\triangleright\triangleright} \geq x \iff x^{\triangleright\triangleright} \wedge x = x$; thus conditions (1) and (1') can be written as equations. For (2) and (2'), we have $x \leq y \implies x^\triangleleft \geq y^\triangleleft$ is equivalent to $(x \wedge y)^\triangleleft \wedge y^\triangleleft = y^\triangleleft$ and $x \leq y \implies x^\triangleright \geq y^\triangleright$ equivalent to $(x \wedge y)^\triangleright \wedge y^\triangleright = y^\triangleright$. Therefore the class of weakly dicomplemented lattices form a variety. We denote it by WDL. Similarly, the class WCL of weakly complemented lattices and the class DCL of dual weakly complemented lattices are varieties. These classes have been introduced to capture the notion of negation on “concepts” [Wi82, Wi96, Wi00, Kw04], based on the work of Boole [Bo54].

The following properties are easy to verify:

- (4) $y \vee y^\triangleleft = 1, 0^\triangleleft = 1, y \wedge y^\triangleright = 0, 1^\triangleright = 0$ and $y^\triangleright \leq y^\triangleleft$ by (3) and (3'),

- (5) $x^{\Delta\Delta\Delta} = x^\Delta$ and $x \mapsto x^{\Delta\Delta}$ is a kernel operator on L as well as $x^{\nabla\nabla\nabla} = x^\nabla$ and $x \mapsto x^{\nabla\nabla}$ is a closure operator on L by (1)–(2').

Before we move to more properties, we give some examples.

- (a) The motivating examples of weakly dicomplemented lattices are concept algebras. These are concept lattices with a weak negation and a weak opposition. For a detailed account on concept algebras, we refer the reader to [Wi00], [Kw04] or [GK07].
- (b) The natural examples of weakly dicomplemented lattices are Boolean algebras. In fact if $(B, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra then $(B, \wedge, \vee, \neg, \neg, 0, 1)$ (the complementation is duplicated, i.e. $x^\Delta := \bar{x} =: x^\nabla$) is a weakly dicomplemented lattice.
- (c) Each bounded lattice can be endowed with a trivial weak dicomplementation by defining $(1, 1)$, $(0, 0)$ and $(1, 0)$ as the dicomplement of 0 , 1 and of each $x \notin \{0, 1\}$, respectively.

Theorem 1. *Weakly complemented lattices are exactly non empty lattices satisfying the equations (1)–(3) in Definition 1.*

Of course, weakly complemented lattices satisfy the equations (1)–(3) in Definition 1. So what we should prove, is that, all lattices satisfying the equations (1) – (3) are bounded.

Proof. Let L be a non empty lattice satisfying the equations (1)–(3'). Let $x \in L$. We set $1 := x \vee x^\Delta$ and $0 := 1^\Delta$. We are going to prove that 1 and 0 are respectively the greatest and lowest element of L . Let y be an arbitrary element of L . We have $1 \geq y \wedge 1 = y \wedge (x \vee x^\Delta) \geq (y \wedge x) \vee (y \wedge x^\Delta) = y$, by (3). Thus $x \vee x^\Delta$ is the greatest element of L . Of course, if L was equipped with a unary operation ∇ satisfying the equation (1')–(3') we could use the same argument as above to say that $x \wedge x^\nabla$ is the smallest element of L . Unfortunately we have to check that $0 := 1^\Delta$ is less than every element of L . So let $y \in L$. We want to prove that $0 \leq y$. Note that $(y \wedge y^\Delta)^\Delta \geq y^\Delta \vee y^{\Delta\Delta} = 1$; thus $(y \wedge y^\Delta)^\Delta = 1$. For an arbitrary element z of L , we have

$$0 \wedge z = 1^\Delta \wedge z = (y \wedge y^\Delta)^{\Delta\Delta} \wedge z \leq y \wedge y^\Delta \wedge z \leq y \wedge z$$

and

$$0 \wedge z^\Delta = 1^\Delta \wedge z^\Delta = (y \wedge y^\Delta)^{\Delta\Delta} \wedge z^\Delta \leq y \wedge y^\Delta \wedge z^\Delta \leq y \wedge z^\Delta.$$

Henceforth $0 = (0 \wedge z) \vee (0 \wedge z^\Delta) \leq (y \wedge z) \vee (y \wedge z^\Delta) = y$. □

2 Axiomatizing Boolean algebras

Definition 2. *A weakly dicomplemented lattice is said to be **with negation** if the unary operations coincide, i.e., if $x^\nabla = x^\Delta$ for all x .*

The class of weakly dicomplemented lattices with negation forms a subvariety of the class of all weakly dicomplemented lattices, denoted by WDN.

Theorem 2. *A weakly dicomplemented lattice $(L, \wedge, \vee, \triangle, \nabla, 0, 1)$ is with negation iff $(L, \wedge, \vee, \triangle, 0, 1)$ and $(L, \wedge, \vee, \nabla, 0, 1)$ are Boolean algebras.*

Proof. We assume that $(L, \wedge, \vee, \triangle, \nabla, 0, 1)$ is a weakly dicomplemented lattice and $\triangle = \nabla$. Recall that $x \vee x^\triangle = 1$ and $x \wedge x^\nabla = 0$. Then with $\triangle = \nabla$, x^\triangle is a complement of x . To prove the distributivity, we will show that the lattices in the variety WDN of weakly dicomplemented lattices with negation are all distributive. To this end it is enough to show that WDN is generated by the two element lattice, i.e every member of WDN with at least three elements is not subdirectly irreducible. We are going to show that for any $L \in \text{WDN}$ with $|L| \geq 3$ there is $\theta_1, \theta_2 \in \text{Con}(L)$ such that $\theta_1 \cap \theta_2 = \Delta$, the trivial congruence (see for example [BS81]).

(i) For $c \in L \setminus \{0, 1\}$, we have $[c, 1] \cong [0, c^\triangle]$. In fact the maps

$$u_{c^\triangle} : [c, 1] \rightarrow [0, c^\triangle] \quad \text{and} \quad v_c : [0, c^\triangle] \rightarrow [c, 1]$$

$$x \mapsto x \wedge c^\triangle \quad \quad \quad x \mapsto x \vee c$$

are order preserving, injective and inverse of each other, since

$$\left. \begin{array}{l} x \wedge c^\triangle = y \wedge c^\triangle \\ x, y \geq c \end{array} \right\} \implies x = (x \wedge c) \vee (x \wedge c^\triangle) = c \vee (y \vee c^\triangle) = y,$$

$$\left. \begin{array}{l} x \vee c = y \vee c \\ x, y \leq c^\triangle \end{array} \right\} \implies x = (x \vee c) \wedge (x \vee c^\nabla) = (y \vee c) \wedge c^\triangle = y,$$

and $v_c \circ u_{c^\triangle}(x) = (x \wedge c^\triangle) \vee c = (x \wedge c^\triangle) \vee (x \wedge c) = x = \text{Id}_{[c, 1]}(x)$ as well as $u_{c^\triangle} \circ v_c(x) = (x \vee c) \wedge c^\triangle = (x \vee c) \wedge (x \vee c^\nabla) = x = \text{Id}_{[0, c^\triangle]}(x)$.

(ii) The maps

$$f_1 : L \rightarrow [c^\triangle, 1] \rightarrow [0, c^{\triangle\triangle}] = [0, c]$$

$$x \mapsto x \vee c^\triangle \mapsto (x \vee c^\triangle) \wedge c = x \wedge c$$

and

$$f_2 : L \rightarrow [c, 1] \rightarrow [0, c^\triangle]$$

$$x \mapsto x \vee c \mapsto (x \vee c) \wedge c^\triangle = x \wedge c^\triangle$$

are lattice homomorphisms. In fact f_1 and f_2 trivially preserve \wedge ; For x, y in L we have, $f_1(x) = u_{c^{\triangle\triangle}}(x \vee c^\triangle)$ and $f_2(x) = u_{c^\triangle}(x \vee c)$. In addition,

$$\begin{aligned} f_1(x \vee y) &= u_{c^{\triangle\triangle}}(x \vee y \vee c^\triangle) = u_{c^{\triangle\triangle}}((x \vee c^\triangle) \vee (y \vee c^\triangle)) \\ &= u_{c^{\triangle\triangle}}(x \vee c^\triangle) \vee u_{c^{\triangle\triangle}}(y \vee c^\triangle) \\ &= f_1(x) \vee f_1(y), \end{aligned}$$

and

$$\begin{aligned} f_2(x \vee y) &= u_{c^\triangle}(x \vee y \vee c) = u_{c^\triangle}((x \vee c) \vee (y \vee c)) \\ &= u_{c^\triangle}(x \vee c) \vee u_{c^\triangle}(y \vee c) \\ &= f_2(x) \vee f_2(y). \end{aligned}$$

(iii) We set $\theta_1 := \ker f_1$ and $\theta_2 := \ker f_2$. Then $\theta_1 \cap \theta_2 = \Delta$. In fact

$$\begin{aligned} (x, y) \in \theta_1 \cap \theta_2 &\implies x \wedge c = y \wedge c \text{ and } x \wedge c^\Delta = y \wedge c^\Delta \\ &\implies x = (x \wedge c) \vee (x \wedge c^\Delta) = (y \wedge c) \vee (y \wedge c^\Delta) = y \\ &\implies (x, y) \in \Delta. \end{aligned}$$

□

Corollary 1. $(L, \wedge, \vee, \Delta, 0, 1)$ is a Boolean algebra iff (L, \wedge, \vee) is a non empty lattice in which $x^{\Delta\Delta} = x$, $x \leq y \implies x^\Delta \geq y^\Delta$ and $(x \wedge y) \vee (x \wedge y^\Delta) = x = (x \vee y) \wedge (x \vee y^\Delta)$.

Theorem 3 (New axiom for Boolean algebras). An algebra $(L, \wedge, \vee, \Delta, 0, 1)$ is a Boolean algebra iff (L, \wedge, \vee) is a non empty lattice in which

$$(x \wedge y) \vee (x \wedge y^\Delta) = (x \vee y) \wedge (x \vee y^\Delta) \text{ for all } x, y \in L \quad (\ddagger).$$

Proof. We are going to show that the equations in Corollary 1 can be derived from (\ddagger) .

- (i) $x \geq (x \wedge y) \vee (x \wedge y^\Delta) = (x \vee y) \wedge (x \vee y^\Delta) \geq x$ implies $(x \wedge y) \vee (x \wedge y^\Delta) = x = (x \vee y) \wedge (x \vee y^\Delta)$.
- (ii) $x = (x \vee y) \wedge (x \vee y^\Delta)$ implies $y \wedge y^\Delta = 0$; thus $x = (x \wedge x^\Delta) \vee (x \wedge x^{\Delta\Delta}) = 0 \vee (x \wedge x^{\Delta\Delta}) = x \wedge x^{\Delta\Delta}$. Hence $x \leq x^{\Delta\Delta}$.
 $x = (x \wedge y) \vee (x \wedge y^\Delta)$ implies $y \vee y^\Delta = 1$; thus $x = (x \vee x^\Delta) \wedge (x \vee x^{\Delta\Delta}) = 1 \wedge (x \vee x^{\Delta\Delta}) = x \vee x^{\Delta\Delta}$. Hence $x \geq x^{\Delta\Delta}$. Therefore $x = x^{\Delta\Delta}$.
- (iii) Let $x \leq y$. Then $x \vee x^\Delta = 1$ implies $y \vee x^\Delta = 1$. Thus $x^\Delta = (x^\Delta \vee y^\Delta) \wedge (x^\Delta \vee y^{\Delta\Delta}) = (x^\Delta \vee y^\Delta) \wedge (x^\Delta \vee y) = x^\Delta \vee y^\Delta$, and $x^\Delta \geq y^\Delta$.

□

In the proof of Theorem 3, we have shown that the conditions (1)–(2') in Definition 1 follow from (3) and (3'), in case $\Delta = \nabla$. Does this hold in general?

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